Statistical mechanics at large charges:

Lessons from AdS/CFT

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Motivation: In the last 7 years there has been substantial progress in developing tools to count microscopic configurations dual to BPS Black holes in AdS/CFT. These developments may be telling us something non-trivial about the inner-workings of the duality, and perhaps, eventually, they could help us to extend it beyond the known frontiers.

In this talk ...

Outline

1. Introduction. Large charge expansion
2. (c,d) - phases of $\mathcal{N} = 4$ SYM on S_3
3. Emergent orbifold quotient
4. Type II B side
5. Exact formula for (c,d) phases: stringy corrections.
6. Conclusions and questions for the future.

Let H be a physical system with discrete spectrum of conserved charges Q (assume the unit of quantization is 1).

Then the number of states in the system is generated by the partition function

$$Z[x] = \mathrm{Tr}_H\, e^{i x Q} = e^{-S[x]}\ .$$

Notation: $\{x\}$ chemical potential dual to $\{Q\}$, $\{e^{i x}\}$ rapidities; $S[x]$ effective action action, $F[x] = -S[x]$ free energy.

They are Fourier integrals :

$$d[Q] = \int_0^{2\pi} Z[x]\, e^{-i\,x\,Q} = \int_0^{2\pi} e^{-S[x]}\, e^{-i\,x\,Q}$$

Question: What properties must $S[x]$ have in order for

$d[Q] \to e^{O(1)\; Q^{\text{Positive Power}}}$ as $Q \to \infty$?

1. S[x] can not be regular. Otherwise...

$$\int_0^{2\pi} e^{-S[x]}\, e^{-i x Q} \to \int_0^{2\pi} e^{-S[x]}\, \delta_{\text{periodic}}[x] = e^{O(1)} \quad \text{for } Q >> 1$$

2. S[x] must have at least one power-like singularity x_{sing} say with power n :

$$S\left[x_{\text{sing}} + \frac{\delta x}{\Lambda}\right] \to \Lambda^n\, s[\delta x]$$

Indeed, defining $Q = q\,\Lambda^{n+1}$ and assuming $\Lambda \to \infty$ it then it follows that:

$$\mathrm{d}[Q] \sim e^{c[q]\, Q^{\frac{n}{n+1}}}$$

$$\sim e^{\Lambda^{n}(-s[\delta x^*] - i\,\delta x^*\, q)}$$

$$\sim e^{-S_{\mathrm{loc}}[x^*] - i\, x^*\, Q}$$

$S_{\mathrm{loc}}[x]$ is the asymptotic expansion of the complete effective action S[x] around the attracting singularity x_{sing}.

$x^* = x^*[Q]$ is infinitely close (aka attracted to) the singularity x_{sing}

Large charge localization

(Take-home message) : A singularity x_{sing} of degree n signals a deconfined phase where entropy grows as

$$\text{Log } d[Q] \sim c[q]\, Q^{\frac{n}{n+1}} \; .$$

This is true provided that:

1 - $[0,1) = \sum_i n_i \Gamma_i$,

where Γ_i are Lefschetz thimbles of $S_{loc}[x] + i\, x\, Q$ and n_i are intersection numbers (integers).

2- $n_i \neq 0$ for a thimble Γ_i intersecting the x^* for which

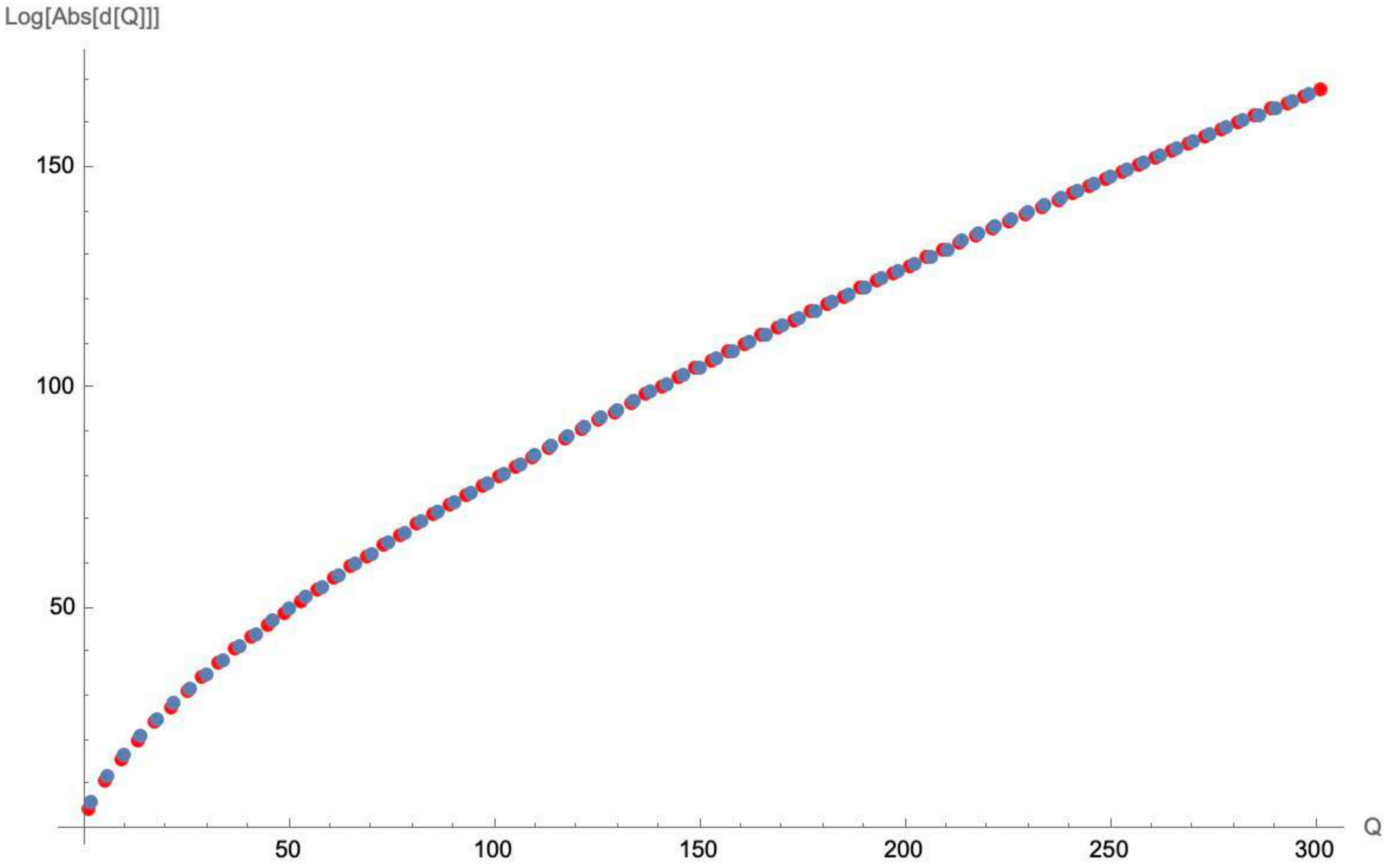
$\mathrm{Re}[S_{loc}[x^*] + i\, x^*\, Q] < 0$ ... otherwise ... no growth!

A toy example (Assume $Q \in \mathbb{Z}_+$)

$$\int_{u^*}^{2\pi+u^*} du\, e^{-\frac{\pi i}{\sin\left[\frac{u}{2}\right]^2} - i u Q} \underset{Q>>1}{\rightarrow} 2 \times \int_{\Gamma_{loc,u^*}} du\, e^{-\frac{4\pi i}{u^2} - i u Q}$$

$$\left| \int_{u^*}^{2\pi+u^*} du\, e^{-\frac{\pi i}{\sin\left[\frac{u}{2}\right]^2} - i u Q} \right| \underset{Q>>1}{\rightarrow} 2 \times \frac{\pi^{2/3}}{\sqrt{3}\ Q^{2/3}}\, e^{\frac{3}{2}\sqrt{3}\ \pi^{1/3}\ Q^{2/3}}$$

In this case the approximation works very well even for small Q



Log[Abs[d[Q]]]
150
100
50
50
100
150
200
250
300
Q

What defines the analytic structure of $S[x]$?

$O^{(1)}_i \quad i = 1, 2, \dots ,$ w/ charges $Q[O^{(1)}_i]$

$O^{(1)}_i * O^{(1)}_i = 0$ Fer-Letters $\qquad (O^{(1)}_i)^n \neq 0 , \quad \forall n > 0,$ Bos-Letters

$$O_{i_1,\dots,i_p} := O_{i_1}{}^{(1)} * \dots * O^{(1)}{}_{i_p}$$

The product is fully antisymmetrized in Fer-letters, and fully symmetrized in Bos-Letters

$$\langle O_{i,\dots} , O_{j\dots} \rangle = \delta_{i\dots,j\dots}$$

What defines the analytic structure of $S[x]$?

$$\mathrm{Tr}_H\,(\pm 1)^F\, e^{i\,x\,Q} := \prod_{j\in \mathrm{Bosons}} \frac{1}{\left(1 - e^{i\,x\,Q[O^{(1)}{}_j]}\right)} \cdot \prod_{j\in \mathrm{Fermions}}\left(1 \pm e^{i\,x\,Q[O^{(1)}{}_j]}\right)$$

$$1\pm X \rightarrow \mathrm{Exp}[\mathrm{Log}[1\pm X]] = -\sum_{l=1}^{\infty} \frac{1}{l}\,(-X)^l$$

$$\mathrm{Tr}_H(\pm 1)^F\, e^{i\,x\,Q} := e^{\sum_{l=1}^{\infty} \frac{1}{l}\, \mathrm{Tr}_{B-\mathrm{Lett}}\, e^{i\,l\,x\,Q} - (\mp 1)^l\, \mathrm{Tr}_{F-\mathrm{Lett}}\, e^{i\,l\,x\,Q}}$$

What defines the analytic structure of $S[x]$?

$$S[x] := \Sigma^{\infty}_{l=1} \frac{1}{l} \left(\Sigma_B e^{i\, l\, x\, Q[B]} - (\mp 1)^l \Sigma_F e^{i\, l\, x\, Q[F]} \right)$$

$$O^{(1)} = (D_{\mu_1} \ldots . D_{\mu_L})\, O^{(1)}{}_{\text{seeds}}$$

$$[J_\mu, D_\mu] = +1\, D_\mu \,, \qquad J_\mu \subset \{\text{Q}\}\,, \qquad \omega_\mu \subset \{\text{x}\}$$

What defines the analytic structure of $S[x]$?

Then the effective action is a sum over rational functions

$$S[x] := \sum_{l=1}^{\infty} \frac{1}{l} \sum_B \frac{e^{i l x Q[B]}}{\prod_\mu (1 - e^{i l \omega_\mu})} - (\mp 1)^l \sum_F \frac{e^{i l x Q[F]}}{\prod_\mu (1 - e^{i l \omega_\mu})}$$

$S[x]$ has power-like singularities at $\frac{\omega_\mu}{2\pi} \to -\frac{c}{d} \bmod 1$

→ Deconfined phases (at roots of unity)?

Notice that not much has been assumed about the physical system ...

4d SU(N) $\mathcal{N} = 4$ SYM on S_3

$Q = \{H, J_1, J_2, R_1, R_2, R_3\}$ even generators

x $= \{\beta, \omega_1, \omega_2, \phi_1, \phi_2, \phi_3\}$ (dual chemical potentials)

$\Theta = \{S, S^*$, other 14 cc. pairs$\}$ odd generators

$$\{S, S^*\} = H - J_1 - J_2 - R_1 - R_2 - R_3 \geq 0$$

4d SU(N) $\mathcal{N}=4$ SYM on S_3

$$I_{S,S^*}[x] := e^{-S[x]} = \int_0^1 \mathrm{d}u_i \, e^{-S_g[x;u]}$$

$$e^{-S_g[x;u]} := \quad \mathrm{Tr}_H \, (-1)^F \, e^{-\beta \{S,S^*\}} \, p^{J_1} \, q^{J_2} \, w_1^{R_1} \, w_2^{R_2} \, w_3^{R_3} \, e^{2\pi i \rho(u)}$$

$$\frac{w_1 \, w_2 \, w_3}{p \, q} = 1 \quad \Leftrightarrow \quad \Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi I n_0$$

Defining the index

Seeds $\subset \{X, \bar{X}, Y, \bar{Y}, Z, \bar{Z}, F_{\mu\nu}, \lambda_{\underline{\pm\pm\pm}}, \ldots\}$

with gauge weights (charges) ρ in the adjoint of SU(N)

$$\{S, S^{*}\} = H - J_1 - J_2 - R_1 - R_2 - R_3 = 0$$

$$[D_\mu, \mathrm{H}] = +1\,\mathrm{H} \quad , \quad [D_\mu, J_\mu] = +1\,J_\mu \;\rightarrow\; [D_\mu, \{S, S^{*}\}] = 0 \quad \mu = 1,2.$$

Defining the index

$$- S[x, u] = \sum_{l=1}^{\infty} \frac{1}{l} \sum_{i \neq j=1} \frac{(1-w_1{}^l)(1-w_2{}^l)(1-w_3{}^l)}{(1-p^l)(1-q^l)} \cos[2\,\pi\, l\, u_{\text{ij}}] + (\rho = 0)$$

For simplicity assume from now on that

$$\text{p} = \text{q} = e^{2\,\pi\, i\, \tau}$$

→ Deconfined phases (at roots of unity)!

$$\tau \to -\frac{d}{c}$$

Entropy at large charges

$$\deg_{c,d}[Q] \to e^{O[1]\,\pi\,\frac{J^{2/3}N^{2/3}}{c}} \qquad J = J_1 + J_2$$

N^2 growth at $N \to \infty$ and large charges $\Longrightarrow$ $J \sim N^2 >> 1$

The $\frac{1}{c}$-suppression

$$- F^{\pm}_{c,d}[x^*_{\pm}, u^*] \qquad \text{(Extremized Entropy functional)}$$

$$= S_{\text{loc,gauge}}\, [x^*_{\pm}, u^*] - 2\,\pi\, i\, (\tau^*)_{\pm}\, J - (\Delta^*)_{\pm} \cdot R$$

$$= \frac{1}{c}\, N^2\, \frac{4\,\pi^3\, i \left(\overline{B}_3\left[\frac{c\,\Delta_1{}^*{}_{\pm}}{2\,\pi\, l}\right] + \overline{B}_3\left[\frac{c\,\Delta_2{}^*{}_{\pm}}{2\,\pi\, l}\right] + \overline{B}_3\left[-\frac{c\,\Delta_2{}^*{}_{\pm} + c\,\Delta_3{}^*{}_{\pm}}{2\,\pi\, l}\right]\right)}{3\,(c\,\tau^*_{\pm} + d)^2} - 2\,\pi\, i\, \tau^*_{\pm}\, J - \Delta^*_{\pm} \cdot R$$

$$\to -\frac{N^2}{c}\,\frac{i\,X^*{}_1\,X^*{}_2\,X^*{}_3}{2\,T^{*2}} - \frac{2\,\pi\,i\,T^*}{c}\,J - \frac{X^*}{c}\cdot R^*$$

$$= -\frac{F^{\pm}{}_{1,0}[x^*{}_{\pm},u^*]}{c}$$

For later purposes

$$T := \frac{1}{(c\,\tau + d)^2} \qquad X_{1,2} := c\,\Delta_{1,2} + 2\,\pi\,i\,r \quad X_3 = \pm\,2\,\pi\,i - X_1 - X_2$$

$$\mathrm{GCD}\,(c,\,d) = \mathrm{GCD}(c,\,r) = 1 \qquad d \sim d + c \quad ,\; r \sim r + c$$

A sum over saddle-points

Keeping all the terms intersected by the original multi-dimensional contour of integration .... the complete degeneracy should be recovered (in principle)

$$\deg[Q] = \left| \sum_{+,-,c,d,r} n_{\text{int}}\, e^{F_{c,d}[(x^*)_\pm, u^*]} \right|$$

$$= \sum_{c,d,r} e^{\mathrm{Re}[F_{c,d}[(x^*)_+ , u^*]]}\, 2\, |\mathrm{Cos}[\mathrm{Im}[F_{c,d}[(x^*)_+ , u^*]]]|$$

$$\underset{Q\to\infty}{\to} e^{\mathrm{Re}[F_{1,0}[(x^*)_+ , 0]]}$$

This formula can be formally improved into an exact formula →
improved description of (c,d)-phases in gauge theory

.... but before going into that ...

let us "uncover" the AdS/CFT dual description of what it has been described so far ...

[2104.13932][2301.00763]

$\mathbb{Z}_c$ -orbifold: Emergent periodicity conditions in the free-energy:

$$\Delta_a \to \Delta_a + \frac{2\pi i}{c} \qquad\qquad \tau \to \tau + \frac{1}{c}$$

Thus the effective partition function (index) only counts operators with (large) charges

$$\{Q_{\text{eff}}\} = c\ \{Q_{\text{UV}}\}$$

where $\{Q_{\text{UV}}\}$ is the BPS spectrum of SU(N) N = 4 SYM on S_3 spacing 1(1/2). Thus $\{Q_{\text{eff}}\}$ has lattice spacing c (c/2).

A geometric understanding of this $\mathbb{Z}_c$- orbifold projection from N = 4 SYM is unclear. In the string theory side though:

The surviving stringy-spectrum comes from an orbifold projection (→) of the complete KK-tower in:

boundary S_3 & internal S_5 → $S_{3,c}$ & $S_{5,c}$

The quotient in the spectrum is bound to be induced by the coordinates identifications

$$\phi_a \sim \phi_a + 2\pi \rightarrow \phi_a \sim \phi_a + 2\pi \frac{r}{c}, \quad r \sim r + c$$

$a = 1, 2, 3, 4, 5$

which "multiplies" the KK spectrum of stringy excitations by a factor of c

This orbifold projection is an emergent feature at large charges and N. The large-N gravitational solution dual to (c,d)-phases should inherit this feature.

At finite temperature the dual solution must be Euclidean and it should be closely related to a “Wick” rotation of the very same Lorentzian black hole accounting for the complete large charge growth in the leading (1,0)-phase.

The BPS condition implies

$$H = J_1 + J_2 + R_1 + R_2 + R_3 \rightarrow c\,(J_1 + J_2 + R_1 + R_2 + R_3) = c\,H$$

which suggests that the correct periodicity condition in Euclidean time for the (c,d)-dual 10d BPS gravitational solution should be enforced by SUSY conditions to be:

$$t_E \sim t_E + \beta_{\mathrm{BH}} \quad \rightarrow \quad t_E \sim t_E + \frac{\beta_{(1,0)}}{c} \quad , \quad \beta_{(1,0)} = \beta_{\mathrm{BH}}$$

How can we test these expectations?

Step 1: Wick rotate the Lorentzian black hole solutions imposing the guessed periodicity conditions (at finite temperature) and compute the action on-shell

Step 2: Take the SUSY limit on the finite temperature answer (BPS limit):

$$\frac{w_1^* \, w_2^* \, w_3^*}{p^* \, q^*} \to 1 \qquad \text{with } \mathrm{Arg}(w^*), \mathrm{Arg}(p^*) \to 0 \text{ and } \mathrm{Arg}(q^*) \to 2\pi i$$

(independence on β?)

Sameer's discussion later today

Step 3: Compare the BPS on-shell action with the field theory prediction.

After these steps the gravitational Euclidean on-shell action reduces to

$$-\frac{N^2}{c}\;\frac{i\,(c\,\Delta_1^*+2\,\pi\,i\,r)\,(c\,\Delta_2^*+2\,\pi\,i\,r)\,(\pm 2\,\pi\,i-c\,\Delta_1^*-c\,\Delta_2^*-4\,\pi\,i\,r)}{2\,(c\,\tau^*+d)^2}$$

$N^2 \sim \frac{1}{G_5}$ ... matching the CFT result

Note: We find that the analogous story is true in AdS_4 / CFT_3 (with very different technical details in the field theory side, Completely analogous details in the gravitational side).

The orbifold solutions can have fixed loci (at the horizon) e.g. for $r = 0$. There can be stringy degrees of freedom localized there (D-branes).

These excitations produce 1/N corrections that must be understood either from the large charge expansion or the formal exact formula (mentioned before).

Consequently a better understanding of the exact formula may be convenient to further decode the “inner-workings” of the string/gauge theory duality.

[1707.05774]
[1811.04107][1812.09613]
[1909.09597][2012.04815][WIP]
Sameer’s discussion later today

Let us go back to the effective action....

$$- S[x, u] = \sum_{l=1}^{\infty} \frac{1}{l} \sum_{i \neq j=1} \frac{(1-w_1{}^{l})(1-w_2{}^{l})(1-w_3{}^{l})}{(1-p^{l})(1-q^{l})} \cos[2\pi l u_{ij}] + (\rho = 0)$$

Quasiperiodicity properties of the holomorphic extension of $e^{-S[x,u]}$ to the complex u-plane :

$$e^{-S[x,u_i + \tau \rho_i]} = \Theta_i[\tau, u]\, e^{-S[\tau,u]}, \quad \Theta_i[x, u_j + \tau \rho_j] = \Theta_i[x, u]$$

Bethe operator

$$\Theta_i[x, u^*] = \prod_{\rho} \left(e^{-\pi i \rho(u) \rho_i}\right) \prod_{I=1,2,3} \theta_0[\rho[u] + \Delta_I,\ \tau]^{\rho_i}$$

This property implies (this can be seen in two ways) an exact fixed point formula

$$e^{-S[\tau]} = \int_0^1 \mathrm{du}\; e^{-S[x,u]} = \sum_{u^*} \frac{e^{-S[x,u^*]}}{H[x,u^*]}$$

$$H = \mathrm{Det}\left(\frac{1}{2\pi i} \frac{\partial \frac{\Theta_a[x,u]}{\Theta_N[x,u]}}{\partial u_b} \right)_{a,b=1,\ldots,N-1}$$

The fixed points are solutions to a Bethe condition

$$u_i^*: \qquad \Theta_i[x, u^*] = (-1)^N \qquad i = 0, \ldots, N-1$$

Subset of solutions that dominate at $N \to \infty$ with charges $\frac{Q}{N^2}$ = finite

Their eigenvalues are :

$$u^*_i = ((c + N\, k_m)\,\tau + (d + c\, k_c + N\, k_e))\ \frac{i}{N} + \frac{(1-N)}{N}\,(c\tau + d)$$

with,

GCD(c,d) = 1 , $\quad$ d ~ d + c (which is the $\mathbb{Z}_c$ symmetry).

The (c , d) ⇒ label conjugacy classes of

$$(\mathbb{Z}_N)^2 = (\mathbb{Z}_N)_e \otimes (\mathbb{Z}_N)_m$$

electric ⊗ magnetic one-form symmetry whose action in encoded in the arbitrary choice of the integers (k_e, k_m).

The on-shell action for the (c,d) fixed points can be evaluated exactly. It contains exact stringy perturbative and non-perturbative corrections that demand physical understanding

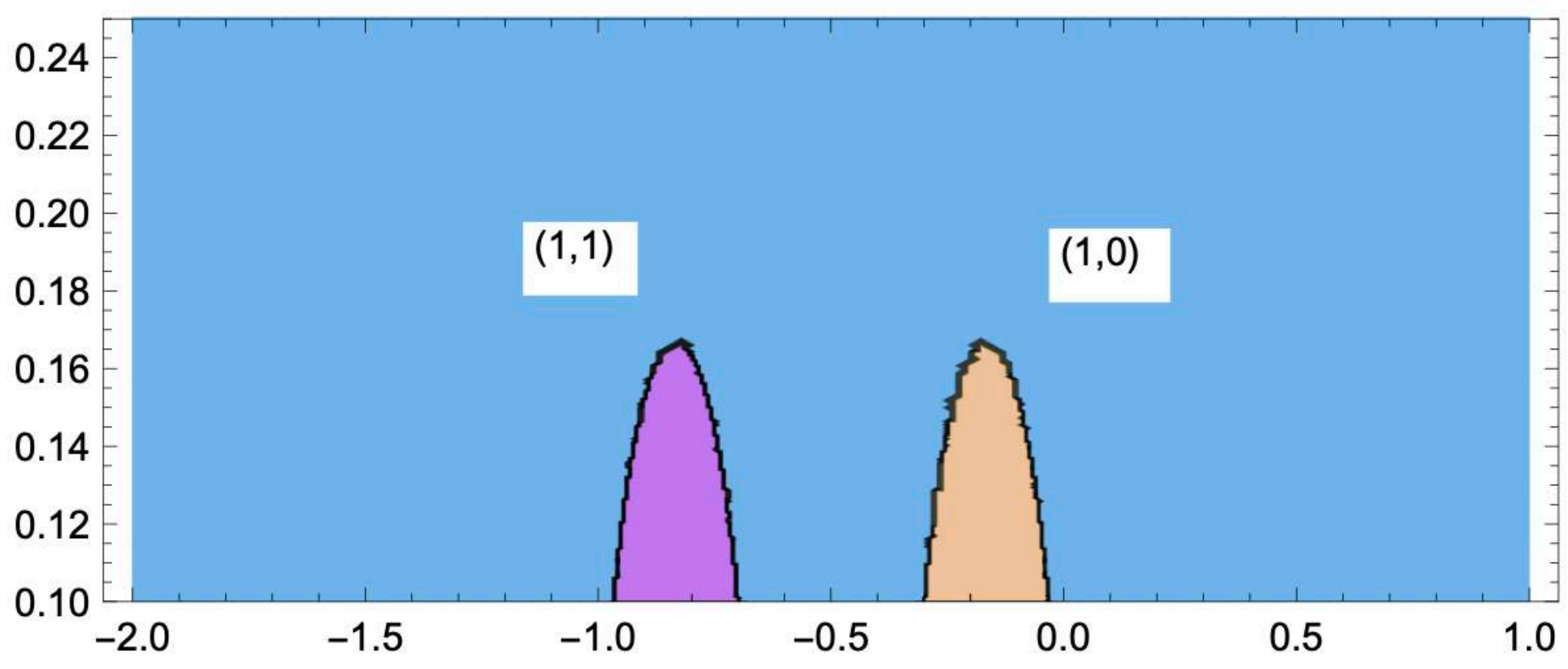
At leading order in $N \to \infty$ and $\tau \to -\frac{d}{c}$ the answer is:

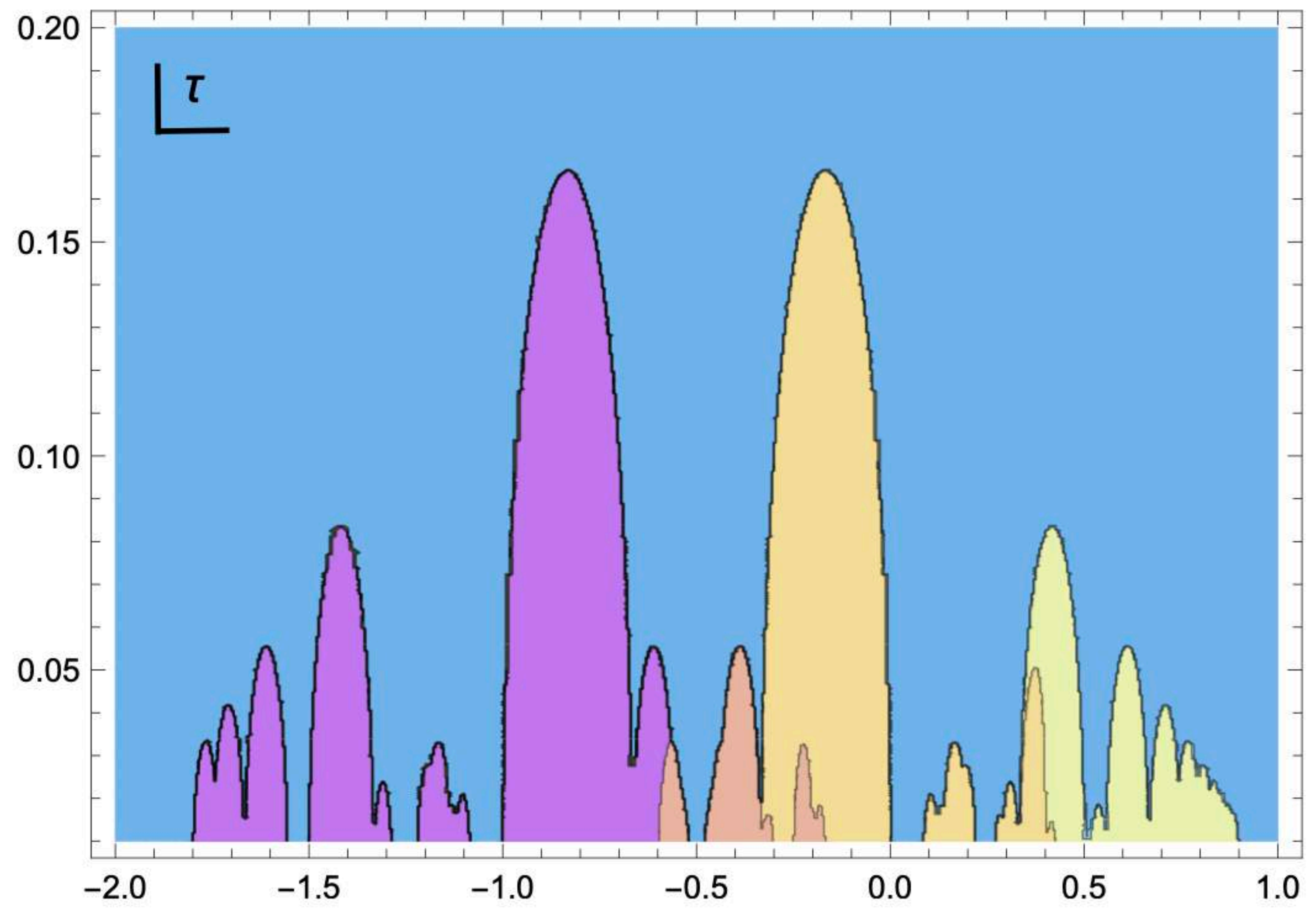
$$S_{c,d}[x] = -\frac{N^2}{c}\,\frac{i\,X_1\,X_2\,X_3}{2\,T^2} + \text{sub}$$

$$S_{0,1}[x] = 0 + \text{sub}$$

which matches the large charge answer before (and gravity).

The phase diagram





[Taken from 1909.09597]

Questions for the future

How much of these features can extend to finite (small) temperature at large charges? Gauge-theory derivation of the PSU(1,1|2) Schwarzian?

Understanding giant graviton expansions at large charges, their relation to the superconformal index and its fixed-point (saddle point) evaluation.

Advance in these questions may give us a better understanding on the "inner-workings" of AdS/CFT and holography.